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Last time:

<u>Theorem</u>

If A is an $n \times m$ matrix and $\vec{x}, \vec{y} \in \mathbb{R}^m$ and k is a scalar then

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$$

and $$A(k\vec{x}) = k(A\vec{x})$$

<u>Cor</u> if $T : \mathbb{R}^m \to \mathbb{R}^n$ is a lin. trans. and $\vec{v}, \vec{w} \in \mathbb{R}^m$, and $a, b \in \mathbb{R}$

then $$\boxed{T(a\vec{v} + b\vec{w}) = aT(\vec{v}) + bT(\vec{w})}. \quad (*)$$ let's call this the linearity property

<u>proof</u>

$$T(a\vec{v} + b\vec{w}) = A(a\vec{v} + b\vec{w}) = A(a\vec{v}) + A(b\vec{w}) = a(A\vec{v}) + b(A\vec{w})$$

$$= aT(\vec{v}) + bT(\vec{w}). \quad //$$

<u>Fact</u> Conversely, if $T : \mathbb{R}^m \to \mathbb{R}^n$ satisfies property $(*)$ then it is a linear transformation. (I.e. if $(*)$ holds then T is really defined by multiplication by some matrix A !!

Why is this true? If we know $(*)$, how do we recover the $n \times m$ matrix A? What info do we need to find it?

<u>Ex</u>

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and let T be the corresp. linear transformation.

Note

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\4\\7\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\5\\8\end{bmatrix} \qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\\9\end{bmatrix}.$$

So since we were given A explicitly, we can rewrite it as

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\left[\,T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\;\middle|\;T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\;\middle|\;T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]$$

Now suppose we only know

(a) T satisfies the linearity property (*)

(b) $T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1\\4\\7\end{bmatrix}$, $T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\5\\8\end{bmatrix}$, $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3\\6\\9\end{bmatrix}$.

How do we find the matrix A and show T is really mult. by A?

I.e. How do we find $T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)$ for any x_1, x_2, x_3?

$$\begin{aligned}
T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) &= T\left(x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= x_1T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)+x_2T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)+x_3T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\\
&= \underbrace{\left[\,T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\;\middle|\;T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\;\middle|\;T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}
\end{aligned}$$

since T is linear

(so you see the importance of the linearity property (*))

Let's give the version of this for $\mathbb{R}^n$.

For convenience, in $\mathbb{R}^n$ we write

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \cdots \quad \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

So in our example

$$A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid T(\vec{e}_3) \right]$$

the version of this in $\mathbb{R}^n$ is essentially Theorem 2.1.2 but we'll extend it a bit.

Theorem

If $T: \mathbb{R}^m \to \mathbb{R}^n$ satisfies the linearity property (*) then it is a linear transformation defined by the $n \times m$ matrix

$$A = \left[T(\vec{e}_1) \mid T(\vec{e}_2) \mid \cdots \mid T(\vec{e}_m) \right]$$

Ex in previous example, knowing T satisfies (*) and knowing

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \quad T(\vec{e}_2) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \quad T(\vec{e}_3) = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$$

forces T to be defined by the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

We didn't have to know A in advance!